Tropical Cyclone Strike Probabilities

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1. Introduction

Tropical cyclones (hurricanes, typhoons, etc.) are violent storms capable of inflicting great destruction. Meteorologists have learned to forecast these storms with superb accuracy. However, no forecast is perfect, and one way to aid interpretation of the forecasts is to compute the probability that a storm will strike a particular location.

This document describes the mathematics used to compute strike probabilities for tropical cyclones. PERL code which implements these equations is available from http://www.solar.ifa.hawaii.edu/Tropical/Bin/StrikeProb.pl. The code can be run from the world wide web at the URL http://www.solar.ifa.hawaii.edu/Tropical/tropical.html.

Given a set of forecast positions and times, the equations below give the probability that the storm center will be within a circular region of radius S about a selectable location, L, at the given forecast times. Cumulative probabilities from one forecast to the next and for the entire forecast period are also computed. See Figure 1 for the geometry.

A number of assumptions are made in the calculation of strike probabilities:

1. The errors in the forecast positions are assumed to be normally distributed.
2. The errors in the forecast positions are assumed to be isotropic.
3. The errors in the forecast positions are assumed to increase with time as the storm position is extrapolated further into the future.
4. The forecasts are assumed to be closely spaced in time (though the code mentioned above can interpolate between forecasts).
5. Plane geometry is assumed.
6. No account is made for the topology of the Earth's surface.

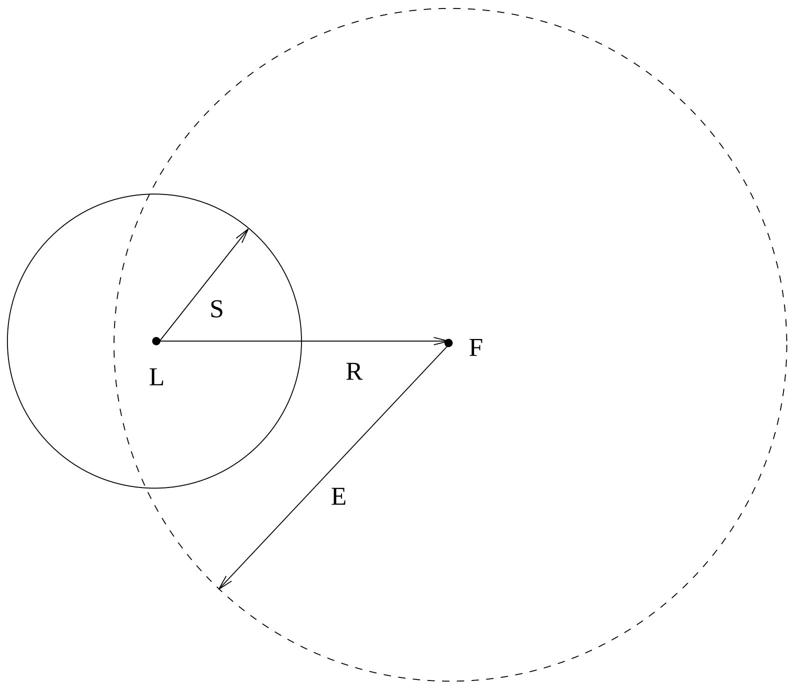

Fig. 1.— Definition of the geometry used in the strike probability calculations. L is the location for which the probabilities are computed. F is the forecast position, R is the distance from L to F, E is the error in the forecast position, and S is the radius of the circular region about L for which the probability is computed.

2. Example

Figures 2 and 3 give the forecast data and the computed strike probabilities for Hamilton, Bermuda during hurricane Erika on 1997 September 8. Figure 3 aids in the interpretation of the raw data shown in Figure 2. In this example, Hamilton is the position L in Figure 1. The forecast positions, F, are listed in Figure 2, and three different values of S are used (60, 120, and 240 nmi). The distance R is the distance from the forecast positions in Figure 2 to Hamilton. The forecast error, E, is $2.55(\Delta t)^{1.18}$, where Δt is the "Relative time: Fcst" in figure 3.

To understand the implications of Figure 3, the first number to examine is the highlighted number in the last row of the "C" columns. These numbers give the cumulative probability of a strike over the entire forecast period (in this case, 1997 Sep 8 09:00 UT through 1997 Sep 11 06:00 UT) in percent. The table of probabilities tells us that there is

Name	Date	Time (UT)	Lat (degrees)	Lon (degrees)	Wind (knots)	Actual / Forecast	WMO
ERIKA	1997-09-08	09:00	22.3N	63.2W	100	ACT	WTNT21
ERIKA	1997-09-08	18:00	23.2N	63.1W	105	FOR	WTNT21
ERIKA	1997-09-09	06:00	25.0N	63.0W	105	FOR	WTNT21
ERIKA	1997-09-09	18:00	27.0N	62.7W	090	FOR	WTNT21
ERIKA	1997-09-10	06:00	29.3N	62.1W	105	FOR	WTNT41
ERIKA	1997-09-11	06:00	33.5N	60.5W	090	FOR	WTNT41

Fig. 2.— The table shows the forecast data used to calculate the strike probabilities. The forecast is for Erika on 1997 Sep 8 at 09:00 UT

a very good chance that Erika will come within 240 nautical miles of Hamilton (80 percent) during the forecast period, but a relatively small chance that Erika will come within 60 nautical miles (8 percent) at some time during the forecast period. It is therefore likely, but not certain, that Hamilton will be spared a direct hit [1].

Looking at the “I” columns, which show the cumulative probabilities from one row to the next, the table also tells us that the storm’s closest approach to Hamilton is most likely to occur sometime on September 10 (universal Time), when the probabilities in the ”I” columns peak. In fact, Erika’s closest approach to Hamilton was about 300 nmi early on September 10 (UT).

The “T” columns give the strike probability at the instant of the forecast. These probabilities are used to calculate the cumulative probabilities in the “I” and “C” columns, but are not generally useful by themselves since they do not refer to a time interval.

The zero hour difference between the “Now” column and the “fcst” column indicates that the strike probability calculation was run just after the forecast was made. The now column shows that the time of closest approach is likely to be 45 to 70 hours away. This helps with the conversion from UT to local time.

The forecast winds at the time of closest approach are in the 90 to 100 knot range. Therefore, this is a strong storm which should be monitored carefully.

[1] Note that a 80 percent probability means that a strike is about 4 times more likely to happen than to not happen. Similarly an 8 percent probability means that a strike is 12 times more likely not to happen that to happen.

Name	Date	Time	Relative Time		60 nmi			120 nmi			240 nmi			Wind
			Now	Fcst	T	I	C	T	I	C	T	I	C	
ERIKA	1997-09-09	18:00 UT	+33 h	+33 h	<1	<1	<1	<1	<1	<1	6	6	6	090
ERIKA	1997-09-10	00:00 UT	+39 h	+39 h	<1	<1	<1	1	1	1	26	28	28	097
ERIKA	1997-09-10	06:00 UT	+45 h	+45 h	1	1	1	8	9	9	46	51	53	105
ERIKA	1997-09-10	18:00 UT	+57 h	+57 h	3	5	5	15	21	21	53	69	76	097
ERIKA	1997-09-11	06:00 UT	+69 h	+69 h	2	6	**8**	10	20	**27**	38	56	**80**	090

Fig. 3.— The table shows an example of the strike probability calculation for Hamilton, Bermuda during hurricane Erika on 1997 Sep 8. The "T" column shows the strike probability at the instant shown: $P(A_i|F)$. The "I" column shows the cumulative probability during the Interval from the previous time shown to the time shown: $P(A_{i-1} + A_i|F)$. The "C" column shows the cumulative probability from the time of the last known storm position to the time shown: $P(A_0 + ... + A_i|F)$. The relative time shows the time difference, in hours, from the time the calculation was run and from the time the forecast was made to the time of the forecast listed.

3. How the Strike Probabilities are Computed

The probability that a storm is a distance x from the forecast position is assumed to be normally distributed:

$$p(x) = \frac{1}{\pi E^2} e^{-x^2/E^2} \tag{1}$$

where E is related to the error in the forecast (Appendix B). The probability is normalized such that

$$\int_0^\infty p(x) 2\pi x dx = 1 \ . \tag{2}$$

The error in the forecast is assumed to increase linearly with time:

$$E = E_0 + s\Delta t \tag{3}$$

where s is a constant, Δt is the difference between the time of the forecast and the time the forecast was made, and E_0 is the error in location at $\Delta t = 0$, i.e. the error in the last observed storm location. With the forcast error increasing with time, later forecast positions are considered less reliable. We take s to be 3.1 nmi/h and E_0 to be 20 nmi (Appendix B). A linear increase of E with time is not required; any functional form could be used. Another good expression is

$$E = E_0 + a(\Delta t)^b \ . \tag{4}$$

Reasonable values are $a = 1.63$ nmi/h and $b = 1.18$. The values of s, E_0, a, and b may differ from storm to storm, from region to region, and from season to season, depending on the accuracy of the forecast models, etc. Appendix B describes how these parameters are computed from an examination of past forecast errors.

The probability that a storm will be within some radius, S, of a specified position, L, at the time of a forecast is given by an integration of $p(x)$ over a circle of radius S centered at L,

$$K(R, S, E) = \int_0^S \int_0^{2\pi} p(x) r d\theta dr \tag{5}$$

where R is the distance from L to the forecast position and x is the distance from the integration position to the forecast position:

$$x^2 = r^2 + R^2 - 2Rr\cos\theta \ . \tag{6}$$

The integration is tedious but straightforward yielding

$$K(R, S, E) = e^{-R^2/E^2} \sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{R}{E}\right)^{2i} P_\gamma\left(i+1, (S/E)^2\right) \tag{7}$$

where P_γ is the incomplete gamma function (see appendix A). The series normally converges in a few terms unless R/E and S/E are $\gg 1$. Note the following special cases:

$$K(R, S=0, E) = 0 \tag{8}$$
$$K(R, S=\infty, E) = 1 \tag{9}$$
$$K(R=0, S, E) = 1 - e^{-S^2/E^2} \tag{10}$$
$$K(R=\infty, S, E) = 0 \tag{11}$$

The expression for $K(R = 0, S, E)$ gives the probability that the storm center will be within a radius S of the forecast position.

It is also useful to compute the probability that the storm is within S of L at some time in the interval between two forecast positions/times. To compute the cumulative probability between forecasts, the probability equation

$$P(A+B|F) = P(A|F) + P(B|F) - P(B|F)P(A|BF) \tag{12}$$

is used. In this notation, $P(X|Y)$ means the probability that hypothesis X is true given the information Y. Here, A represents the hypothesis that the storm is within a radius S of some position, L, at the time of forecast A and similarly for B. Equation (12) assumes that the time between forecast positions is sufficiently short that the probability at the times of A and B are similar. $P(A|F)$ is the probability of A given the forecast information F and similarly for $P(B|F)$. F represents the entire set of forecast positions and times. In the last term, $P(A|BF)$ is the probability that A is true given that B is true. In other words, the probability that the storm is within a radius S of position L at the time of forecast A given the hypothetical situation that the storm was within a radius S of position L at the time of forecast B. The terms are computed as follows:

$$P(A|F) = K(R_A, S, E_A) \tag{13}$$

$$P(B|F) = K(R_B, S, E_B) \tag{14}$$

$$P(A|BF) = \hat{K}(A|B) \tag{15}$$

where E_A is $s\Delta t_A$, R_A is the distance from forecast position A to the requested position and similarly for E_B and R_B. The calculation of $P(A|BF)$ requires some discussion and is left to the appendix.

The calculation of $P(A|BF)$ in the appendix assumes that forecast B is before forecast A. Below we will need to compute $P(B|AF)$, the probability of a strike at the time of some forecast given that a strike occurred at the time of some later forecast. This probability is computed from Bayes' theorem:

$$P(B|AF) = P(A|BF)\frac{P(B|F)}{P(A|F)} \; . \tag{16}$$

We now proceed to generalize the above calculation to give the cumulative probability over many forecast positions/times. This cumulative probability over an extended forecast period is computed by repeatedly using equation (12). This is a recursive procedure. For this calculation, S_i represents the hypothesis that the storm was located within a radius S of L at some time before forecast i (here, the forecasts are indexed by i). A_i represents the hypothesis that the storm is within S of L at the time of forecast i. $P(S_i|F)$ is built up by repeated application of equation (12):

$$P(S_i|F) = P(S_{i-1}|F) + P(A_i|F) - P(A_i|F)P(S_{i-1}|A_iF) \; , \tag{17}$$

with

$$P(S_0|F) = P(A_0|F) \ . \tag{18}$$

$P(S_{i-1}|A_iF)$ is also built up by repeated application of equation (12)

$$P(S_{i-1}|A_iF) = P(S_{i-2}|A_iF) + P(A_{i-1}|A_iF) - P(A_{i-1}|A_iF)P(S_{i-2}|A_{i-1}A_iF) \ , \tag{19}$$

with

$$\begin{aligned} P(A_j|A_iF) &= \frac{P(A_j|F)}{P(A_i|F)}\hat{K}(A_i|A_j) \ , \ j < i \\ P(A_j|A_iF) &= \hat{K}(A_j|A_i) \ , \ j > i \end{aligned} \tag{20}$$

and

$$P(S_0|A_iF) = P(A_0|A_iF) \ . \tag{21}$$

This process is continued by computing $P(S_{i-2}|A_{i-1}A_iF)$ as

$$\begin{aligned} P(S_{i-2}|A_{i-1}A_iF) &= P(S_{i-3}|A_{i-1}A_iF) \\ &+ P(A_{i-2}|A_{i-1}A_iF) \\ &- P(A_{i-2}|A_{i-1}A_iF)P(S_{i-3}|A_{i-2}A_{i-1}A_iF) \ , \end{aligned} \tag{22}$$

etc. This procedure continues until we get to terms like

$$P(S_0|A_2A_3...A_iF) = P(A_0|A_2A_3...A_iF) \tag{23}$$

and

$$P(A_{k-1}|A_kA_{k+1}...A_iF) \ , \ \ 1 \leq k < i \ , \tag{24}$$

which are solved recursively as

$$\begin{aligned} P(A_j|A_kA_{k+1}...A_iF) &\approx P(A_j|A_iF) \\ &+ P(A_j|A_kA_{k+1}...A_{i-1}F) \\ &- P(A_j|A_iF)P(A_i|A_kA_{k+1}...A_{i-1}F) \end{aligned} \tag{25}$$

with $j < k < i$. This decomposition of $P(A_j|A_kA_{k+1}...A_iF)$ continues until we reach terms which can be computed from equation (20).

Equation (25) is not a fundamental identity. Rather, it is motivated by considering the extreme values of $P(A_i|A_kA_{k+1}...A_{i-1}F)$: 0 and 1. If $P(A_i|A_kA_{k+1}...A_{i-1}F)$ is 1, then A_i carries no information that is not already contained in $A_kA_{k+1}...A_{i-1}F$ and we must have $P(A_j|A_kA_{k+1}...A_iF) = P(A_j|A_kA_{k+1}...A_{i-1}F)$. At the other extreme, when $P(A_i|A_kA_{k+1}...A_{i-1}F)$ approaches 0, A_iF tends towards mutual exclusivty with

$A_kA_{k+1}...A_{i-1}F$; if $P(A_i|A_kA_{k+1}...A_{i-1}F)$ were exactly 0, then A_iF and $A_kA_{k+1}...A_{i-1}F$ would be mutually exclusive and $P(A_j|A_kA_{k+1}...A_iF)$ would be undefined. As mutual exclusivity between A_iF and $A_kA_{k+1}...A_{i-1}F$ is approached, we set $P(A_j|A_kA_{k+1}...A_iF)$ to $P(A_j|A_iF) + P(A_j|A_kA_{k+1}...A_{i-1}F)$ under the assumption that, if forecasts i and $k, k+1, ..., i-1$ are so far removed from one another that A_iF and $A_kA_{k+1}...A_{i-1}F$ are nearly mutually exclusive, then $P(A_j|A_iF)$ and $P(A_j|A_kA_{k+1}...A_{i-1}F)$ can be treated independently. Equation (25) gives a linear transition between the two extremes of $P(A_i|A_kA_{k+1}...A_{i-1}F)$.

Finally, we give an approximation for $P(S_i|F)$ which bypasses the more general, but computationally demanding, solution presented above. In practice, this approximation is usually good. Using Bayes' Theorem, equation (19) can be rewritten as

$$P(S_{i-1}|A_iF) = P(S_{i-2}|A_iF) + P(A_{i-1}|A_iF) - P(S_{i-2}|A_iF)P(A_{i-1}|S_{i-2}A_iF) \ . \tag{26}$$

Here $P(A_{i-1}|S_{i-2}A_iF)$ represents the probability the the storm is within S of L at the time of the $i-1$ forecast, given that the storm is within S of L at the time of the next forecast (A_i) *and* sometime before the $i-1$ forecast (S_{i-2}). Since, in this hypothetical situation, the storm is within S of L both before and after the $i-1$ forecast, it must be that $P(A_{i-1}|S_{i-2}A_iF)$ is close to 1.

Setting $P(A_{i-1}|S_{i-2}A_iF) \approx 1$, equation (19) becomes

$$P(S_{i-1}|A_iF) \approx P(A_{i-1}|A_iF) \ , \tag{27}$$

and

$$P(S_i|F) \approx P(S_{i-1}|F) + P(A_i|F) - P(A_i|F)P(A_{i-1}|A_iF) \ . \tag{28}$$

Though approximate, this expression is considerably less demanding computationally than the more general equation (19).

A condition for the cumulative probabilities is that

$$P(S_i|F) \geq P(A_i + A_{i-1}|F) \tag{29}$$

or, if approximation (27) holds, the more strict

$$P(S_i|F) \geq P(S_{i-1}|F) + P(A_i + A_{i-1}|F) - P(A_{i-1} + A_{i-2}|F) \ . \tag{30}$$

These should be checked since the calculation of $P(A|BF)$ is approximate.

A. Details

A.1. Computing $K(R,S,E)$

The integral for $K(R,S,E)$ is solved as follows.

$$\begin{aligned} K(R,S,E) &= \int_0^S \int_0^{2\pi} p(x) r d\theta dr & \text{(A1)} \\ &= \frac{1}{\pi E^2} \int_0^S \int_0^{2\pi} e^{-x^2/E^2} r d\theta dr & \text{(A2)} \\ &= \frac{1}{\pi E^2} \int_0^S \int_0^{2\pi} e^{-(r^2+R^2-2Rr\cos\theta)/E^2} r d\theta dr & \text{(A3)} \\ &= \frac{1}{\pi E^2} e^{-R^2/E^2} \int_0^S \int_0^{2\pi} e^{-r^2/E^2} e^{2Rr\cos\theta/E^2} r d\theta dr & \text{(A4)} \end{aligned}$$

The θ integral is solved by expanding $e^{2Rr\cos\theta/E^2}$ to

$$e^{2Rr\cos\theta/E^2} = \sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{2Rr\cos\theta}{E^2} \right)^i . \quad \text{(A5)}$$

The integrals of the odd powers of $\cos\theta$ are zero, hence, using

$$\int_0^{2\pi} \cos^{2i}\theta d\theta = 4 \frac{\sqrt{\pi}}{2} \frac{\Gamma(i+\frac{1}{2})}{\Gamma(i+1)} , \quad \text{(A6)}$$

$$K(R,S,E) = \frac{1}{\pi E^2} e^{-R^2/E^2} \int_0^S r e^{-r^2/E^2} \sum_{i=0}^{\infty} \left(\frac{2Rr}{E^2} \right)^{2i} \frac{1}{(2i)!} \frac{1}{\Gamma(i+1)} \frac{\Gamma(i+\frac{1}{2})}{\sqrt{\pi}} 2\pi dr . \quad \text{(A7)}$$

Switching the order of summation and integration yields,

$$K(R,S,E) = \frac{2}{E^2} e^{-R^2/E^2} \sum_{i=0}^{\infty} \frac{1}{(2i)!} \frac{1}{\Gamma(i+1)} \frac{\Gamma(i+\frac{1}{2})}{\sqrt{\pi}} \left(\frac{2R}{E^2} \right)^{2i} \int_0^S e^{-r^2/E^2} r^{2i+1} dr . \quad \text{(A8)}$$

The r integral is solved as

$$\begin{aligned} \int_0^S e^{-r^2/E^2} r^{2i+1} dr &= \frac{1}{2} E^{2(i+1)} \int_0^{S^2/E^2} e^{-v} v^i dv & \text{(A9)} \\ &= \frac{1}{2} E^{2(i+1)} \gamma\left(i+1, (S/E)^2\right) & \text{(A10)} \end{aligned}$$

Here, $\gamma(a,x)$ is related to the incomplete gamma function, $P_\gamma(a,x)$, by

$$\gamma(a,x) = \Gamma(a) P_\gamma(a,x) , \quad \text{(A11)}$$

where $\Gamma(a)$ is the gamma function.

Combining equation (A10) with equation (A8), and using

$$\frac{2^{2i}\Gamma(i+\frac{1}{2})}{(2i)!\sqrt{\pi}} = \frac{1}{i!} \ , \qquad \text{(A12)}$$

we arrive at equation (7), after some manipulation.

A.2. Computing $P(A|BF)$

The correct method to compute $P(A|BF)$ is to re-forecast the storm position at A for many starting positions within S of L at the time of forecast B, and average K over these new forecasts. This is clearly impractical since it would require running the forecast models for a large number of fictitious storm positions. There are a number of approximations which can replace the correct calculation, however. The three approximations given below are ordered from most accurate, and most computationally demanding, to least accurate, and least computationally demanding.

There are several conditions which $P(A|BF)$ should satisfy. Since the calculation of $P(A|BF)$ is approximate, these conditions should be checked. When these conditions are violated, it generally means that the forecasts are too far apart and should be further interpolated. Since $P(A+B|F) = P(A|F) + P(B|F) - P(B|F)P(A|BF)$ must not exceed 1, but must be greater than both $P(A|F)$ and $P(B|F)$, we have

$$\frac{P(A|F)}{P(B|F)} \geq P(A|BF) \geq \frac{P(A|F) + P(B|F) - 1}{P(B|F)} \ . \qquad \text{(A13)}$$

If the criteria are violated, the forecasts should be interpolated with a finer time resolution.

We also note that, when A and B are very close in time, $P(A|BF)$ should tend to 1, and, when $P(B|F)$ approaches 1, $P(A|BF)$ should approach $P(A|F)$. Also note that, as defined below, the calculation assumes that forecast B is before A. If this is not the case, Bayes' Theorem (equation 16) should be used.

A.2.1. Approximation 0

This approximation assumes that the direction and distance from B to A remains the same as the forecast, but the location of B is shifted to L when computing $P(A|BF)$. In

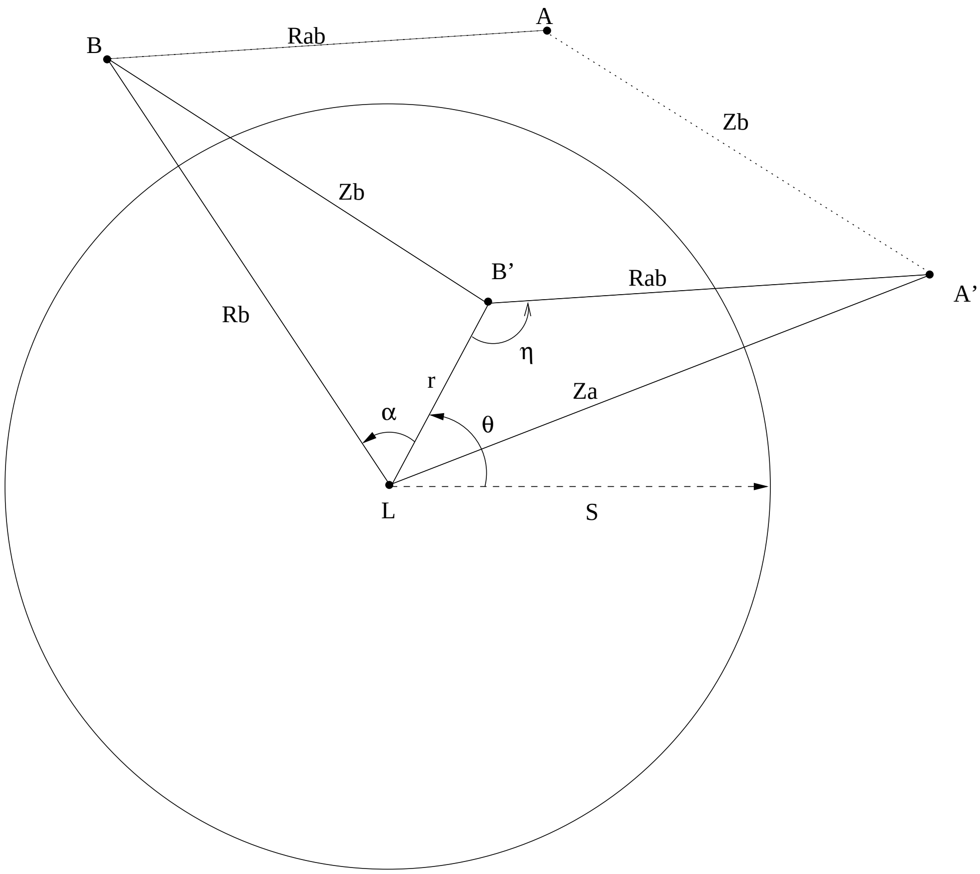

Fig. 4.— Definition of the geometry used in Approximation 0. Forecast B is moved to locations B' within S of L while the distance and direction from B' to the next forecast, A', is kept the same as from B to A.

this case, the forecast is shifted to account for the assumed truth of B and $P(A|BF)$ is given by

$$\hat{K_0}(A|B) = \frac{\int_0^S \int_0^{2\pi} e^{-Z_b^2/E_b^2} K(Z_a, S, E_{ab}) r d\theta dr}{\int_0^S \int_0^{2\pi} e^{-Z_b^2/E_b^2} r d\theta dr} \tag{A14}$$

where $Z_a^2 = r^2 + R_{ab}^2 - 2rR_{ab}\cos\eta$, R_{ab} is the distance between A and B, $Z_b^2 = r^2 + R_b^2 - 2rR_b \cos\alpha$, R_b is the distance between B and L, α is $\angle BLB'$, η is $\angle LB'A'$, E_b is $s\Delta t_b$, and E_{ab} is $s(\Delta t_a - \Delta t_b)$. Here we make the approximation that if B is true, then the storm is somewhere within S of L at the time of forecast B with liklihood $p(B')$ at each point B' within S of L. $p(B')$ is the probability that the storm is at B' given the forecast position of B: $\frac{1}{\pi E_b^2} e^{-R_B^2/E_b^2}$.

The denominator of equation (A14) is $\pi E_b^2 K(R_b, S, E_b) = \pi E_b^2 P(B|F)$, so

$$\hat{K}_0(A|B) = \frac{1}{\pi E_b^2 P(B|F)} \int_0^S \int_0^{2\pi} e^{-Z_b^2/E_b^2} K(Z_a, S, E_{ab}) r \, d\theta \, dr \tag{A15}$$

$$= \frac{1}{\pi E_b^2 P(B|F)} \sum_{i=0}^{\infty} \beta_i \left(S, E_{ab}\right) \int_0^S \int_0^{2\pi} e^{-Z_b^2/E_b^2} e^{-Z_a^2/E_{ab}^2} \left(\frac{Z_a}{E_{ab}}\right)^{2i} r \, d\theta dr \tag{A16}$$

where

$$\beta_i(S, E) \equiv \frac{1}{i!} P_\gamma \left(i + 1, (S/E)^2\right) \ . \tag{A17}$$

$\hat{K}_0(A|B)$ assumes that forecast B is before forecast A. If this is not the case, Bayes' Theorem (equation 16) should be used.

Note that $\hat{K}_0$ tends to 1 when A and B are very close in space and time, i.e. when R_{AB}/E goes to zero and S/E_{ab} goes to infinity. Thus, the cumulative probabilities will not depend on the details of the forecast interpolation scheme.

Further, when $P(B|F)$ approaches 1, $\hat{K}_0$ tends to $P(A|F)$. This is easily seen by writing $K(Z_a, S, E_{ab})$ as $\int_0^S \int_0^{2\pi} e^{-Z_a^2/E_{ab}^2} r' d\theta' dr'$. When $S \gg E_b$ and $R_b \ll E_b$, the numerator of equation (A14) becomes the convolution of two Gaussians, and $\hat{K}_0(A|B)$ goes to $P(A|F)$, since $E_b^2 + E_{ab}^2 = E_a^2$ when $E_a \gg 2E_b E_{ab}$.

As defined above, $P(A|BF)$ satifies the condition that $P(A|BF) < P(A|F)/P(B|F)$ since, as $P(B|F)$ approaches 1, $\hat{K}_0$ tends to $P(A|F)$ from below and the numerator is bounded above by $P(A|F)$. The condition that $P(A|BF) > (P(A|F) + P(B|F) - 1)/P(B|F)$ is only violated when the forecasts are not interpolated to a fine enough resolution.

This is the only approximation given here which satifies both properties that $\hat{K}_0$ tends to 1 when A and B are very close in space and time, and, when $P(B|F)$ approaches 1, $\hat{K}_0$ tends to $P(A|F)$. It is, hence, the best approximation. Unfortunately, the integration in the numerator has not yet yielded to any simplification. It can be solved numerically, but the double integral is very demanding and may not be practical. Approximation 1, below, makes the further approximation that $p(B') \approx$ constant, i.e. $E_b \gg S$ or $P(B|F) \ll 1$, which is more tractable.

A.2.2. Approximation 1

This approximation assumes that the direction and distance from B to A remains the same as the forecast, but the location of B is shifted to L when computing $P(A|BF)$. In this case, the forecast is shifted to account for the assumed truth of B and $\hat{K}_1$ $(= P(A|BF))$

is given by

$$\hat{K}_1(A|B) = \frac{1}{\pi S^2}\int_0^S\int_0^{2\pi} K(z,S,E) r d\theta dr \tag{A18}$$

where $z^2 = r^2 + R_{AB}^2 - 2rR_{AB}\cos\theta$, R_{AB} is the distance between A and B, and E is $s(\Delta t_A - \Delta t_B)$. Here we make the approximation that if B is true, then the storm is somewhere within S of L at the time of forecast B with equal liklihood everywhere within S of L.

Note that $\hat{K}_1(A|B)$ assumes that forecast B is before forecast A. If this is not the case, Bayes' Theorem (equation 16) should be used.

Inserting the expression for $K(z,S,E)$ into equation (A18), we have

$$\hat{K}_1(A|B) = \frac{1}{\pi S^2}\sum_{i=0}^{\infty}\frac{1}{i!}P_\gamma\left(i+1,\left(\frac{S}{E}\right)^2\right)\int_0^S\int_0^{2\pi} e^{-z^2/E^2}\left(\frac{z}{E}\right)^{2i} r\,d\theta\,dr\ . \tag{A19}$$

Although an analytical solution is not possible, the double integral can be converted to a single integral which is easy to compute numerically.

First, remove the constant E, since it just gets in the way. Let $\hat{r} = r/E$, $\hat{u} = z/E$, $\hat{S} = S/E$, and $\hat{R} = R/E$. Then the integral becomes

$$E^2\int_0^{\hat{S}}\int_0^{2\pi} e^{-\hat{u}^2}\,(\hat{u})^{2i}\,\hat{r} d\theta\,d\hat{r}, \tag{A20}$$

where $\hat{u}^2 = \hat{r}^2 + \hat{R}^2 - 2\hat{r}\hat{R}\cos\theta$.

Let's now forget about E altogether, and just focus on the integral:

$$\int_0^S\int_0^{2\pi} e^{-u^2}u^{2i}\,r d\theta\,dr, \tag{A21}$$

with $u^2 = r^2 + R^2 - 2rR\cos\theta$; where we have to keep in mind that the various quantities have been redefined. The quantity u measures the distance between the points $(R,0)$ and (r,θ).

Let $P(u) = e^{-u^2}u^{2i}$, so that the integral may be recognized as the volume above the circle C (center $(0,0)$ and radius S), and beneath the surface $z = P(u(r,\theta))$. The volume is calculated using the polar element of area: $rd\theta\,dr$.

Referring to Figure 5, the new coordinate system will be the polar coordinates (ρ,ϕ) associated with the center $(R,0)$, and the polar element of area $\rho d\phi\,d\rho$.

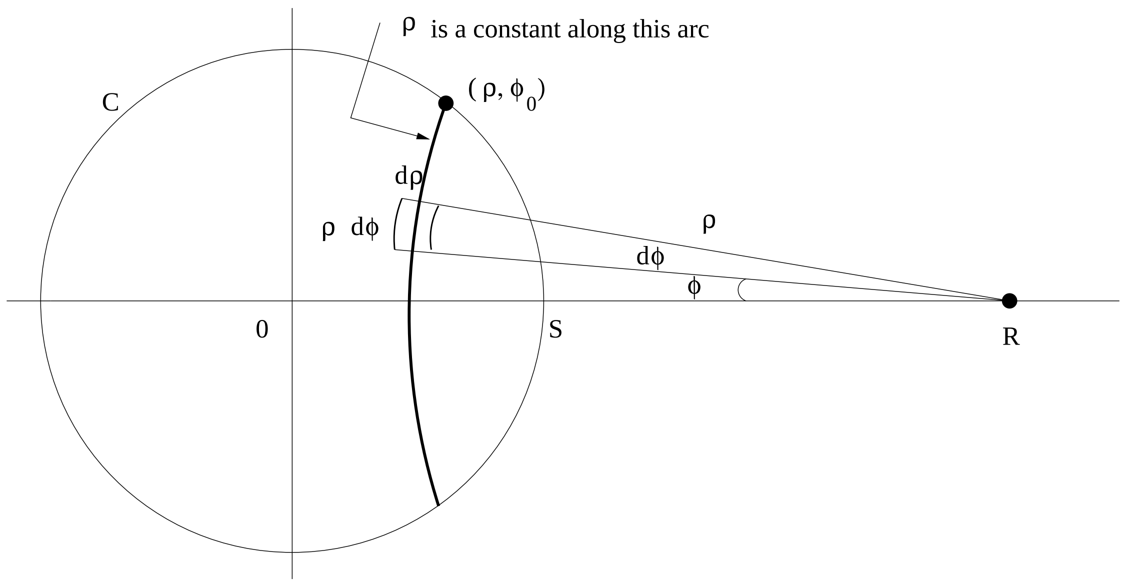

Fig. 5.— Definition of the geometry used in the integration.

The integral then becomes

$$\int_{R-S}^{R+S} \int_{-\phi_0}^{\phi_0} P(\rho)\, \rho d\phi\, d\rho = \int_{R-S}^{R+S} \rho\, P(\rho) \int_{-\phi_0}^{\phi_0} d\phi\, d\rho = 2 \int_{R-S}^{R+S} \rho\, P(\rho)\, \phi_0(\rho)\, d\rho. \qquad \text{(A22)}$$

The calculation of $\phi_0(\rho)$ is simple geometry. The point (ρ, ϕ_0) is one of the intersection points of the two circles

$$(x - R)^2 + y^2 = \rho^2 \qquad \text{and} \qquad x^2 + y^2 = S^2. \qquad \text{(A23)}$$

The solution is given by:

$$x = \frac{R^2 + S^2 - \rho^2}{2R} \qquad \text{and} \qquad y = \sqrt{S^2 - x^2}, \qquad \text{(A24)}$$

leading to

$$\cos\left(\phi_0(\rho)\right) = \frac{R^2 - S^2 + \rho^2}{2R\rho}. \qquad \text{(A25)}$$

A final form for the integral is then

$$2 \int_{R-S}^{R+S} \rho\, \cos^{-1}\left[(R^2 - S^2 + \rho^2)/(2R\rho)\right]\, P(\rho)\, d\rho, \qquad \text{(A26)}$$

or

$$2\int_{R-S}^{R+S} \rho^{2i+1}\ \cos^{-1}\Big[(R^2 - S^2 + \rho^2)/(2R\rho)\Big]\ e^{-\rho^2}\ d\rho. \tag{A27}$$

In the course of solving this integral numerically, when $R + \rho < S$ the $\cos^{-1}$ term is set to π. This only happens when R is less than S. Also, when R is less than S, the lower bound on the integral must be set to 0 rather than $R - S$, since the $\cos^{-1}$ term is set to π.

Finally, re-introducing the E's and reverting to the original definitions of R and S, we have

$$\hat{K}_1(A|B) = \frac{2E^2}{\pi S^2}\sum_{i=0}^{\infty} P_\gamma\left(i+1, \left(\frac{S}{E}\right)^2\right)\int_{\frac{R-S}{E}}^{\frac{R+S}{E}} \frac{\rho^{2i+1}}{i!}\ \cos^{-1}\Big[(R^2 - S^2 + E^2\rho^2)/(2R\rho)\Big]\ e^{-\rho^2}\ d\rho\ . \tag{A28}$$

When computing the integral numerically, it is useful to rewrite

$$\frac{\rho^{2i+1}}{i!}e^{-\rho^2} \tag{A29}$$

as

$$e^{(2i+1)\ln(\rho) - \ln(i!) - \rho^2}\ . \tag{A30}$$

$\ln(i!)$ can be efficiently computed as $\ln(\Gamma(i+1))$ using standard numerical techniques. A variable spacing in the numerical integration is essential: most of the variation in the integrand occurs at small ρ. At large ρ, the exponential term dominates and a larger spacing can be tolerated.

Finally, note that $\hat{K}_1(A|B)$ tends to 1 when A and B are very close in space and time, i.e. when R/E goes to zero and S/E goes to infinity. Thus, the cumulative probabilities will not depend on the details of the forecast interpolation scheme. In this special case, the integral can be done analytically and equation (A19) becomes

$$\hat{K}_1 = \frac{E^2}{S^2}\sum_{i=0}^{\infty} P_\gamma\left(i+1, \left(\frac{S}{E}\right)^2\right)^2 \tag{A31}$$

for $R_{AB}/E = 0$, and,

$$\hat{K}_1 = 1 \tag{A32}$$

for $R_{AB}/E = 0$ and $S/E = \infty$. The series converges very slowly in this case, so care must be taken when S/E goes to infinity not to truncate the summation prematurely when applying equation (A28).

A.2.3. Approximation 2

Alternatively one could assume that forecast A and forecast B are very closely spaced in time so that the storm does not evolve from B to A. In this case, $\hat{K}_2$ gives the probability that the storm remains within S of L at the time of forecast A, given that it is somewhere within S of L at the time of forecast B. This is an approximation valid when the time between forecasts is not too large.

In this case, $\hat{K}_2$ is defined as

$$\hat{K}_2(A|B) = \frac{1}{\pi S^2} \int_0^S K(r, S, E) 2\pi r dr \ , \tag{A33}$$

where E is $s(\Delta t_A - \Delta t_B)$.

As with Approximation 1, $\hat{K}_2(A|B)$ assumes that forecast B is before A. If this is not the case, Bayes' Theorem should be used.

The integral for $\hat{K}_2$ is similar to the integral for K.

$$\begin{aligned}
\hat{K}_2 &= \frac{1}{\pi S^2} \int_0^S K(r, S, E) 2\pi r dr & \text{(A34)} \\
&= \frac{1}{\pi S^2} \int_0^S \sum_{i=0}^{\infty} \beta_i(S, E) e^{-r^2/E^2} \left(\frac{r}{E}\right)^{2i} 2\pi r dr & \text{(A35)} \\
&= \frac{E^2}{S^2} \int_0^{S^2/E^2} \sum_{i=0}^{\infty} \beta_i(S, E) e^{-y} y^i dy & \text{(A36)} \\
&= \frac{E^2}{S^2} \sum_{i=0}^{\infty} \beta_i(S, E) \int_0^{S^2/E^2} e^{-y} y^i dy & \text{(A37)} \\
&= \frac{E^2}{S^2} \sum_{i=0}^{\infty} \beta_i(S, E) \gamma\left(i+1, (S/E)^2\right) & \text{(A38)} \\
&= \frac{E^2}{S^2} \sum_{i=0}^{\infty} \frac{1}{i!} \left[P_\gamma\left(i+1, (S/E)^2\right)\right]^2 \Gamma(i+1) & \text{(A39)} \\
&= \frac{E^2}{S^2} \sum_{i=0}^{\infty} \left[P_\gamma\left(i+1, (S/E)^2\right)\right]^2 & \text{(A40)}
\end{aligned}$$

Above, $\beta_i(S, E)$ is

$$\beta_i(S, E) = \frac{1}{i!} P_\gamma\left(i+1, (S/E)^2\right) \ . \tag{A41}$$

Putting this all together, we have

$$\hat{K}_2(A|B) = \frac{E^2}{S^2} \sum_{i=0}^{\infty} \left[P_\gamma\left(i+1, (S/E)^2\right)\right]^2 \ . \tag{A42}$$

For this second approximation to be valid, the forecasts must be very closely spaced and require interpolation. Further, we sometimes require that A and B not be adjacent forecasts, violating the condition of the approximation.

A.2.4. Approximation 3

Finally, one could assume that B has no effect on A. In this case $\hat{K_3}$ is simply $P(A|F)$. This is the least demanding assumption computationally since it makes no assumptions about the how the forecast would change if B were true. In the absence of any knowledge of how A will be affected when B is true, this may be the best choice, though this is rarely the case. In any event, this assumption will tend to overestimate the strike probabilities and should only be used when there is truly no information on how B affects A. Also note that this approximation violates the condition that $P(A|BF)$ tends to unity as the time and distance between A and B goes to zero. Hence, the computed probabilities under Approximation 3 will depend on the details of the forecast interpolation. Approximation 3 is strictly correct only when $P(B|F) = 1$.

B. Measuring E

Equation (3) is calibrated by gathering information on forecast errors as a function of the duration of the extrapolation (Δt). At each Δt, the mean error is computed and compared to

$$< x > = \frac{\int_0^\infty x e^{-x^2/E^2} 2\pi x dx}{\int_0^\infty e^{-x^2/E^2} 2\pi x dx} \tag{B1}$$

$$= \frac{\int_0^\infty e^{-x^2/E^2} x^2 dx}{\int_0^\infty e^{-x^2/E^2} x dx} \tag{B2}$$

$$= \frac{E^3 \frac{\sqrt{\pi}}{4}}{E^2 \frac{1}{2}} \tag{B3}$$

$$= E\frac{\sqrt{\pi}}{2} . \tag{B4}$$

Thus, using the measured value of $< x >$ at each value of Δt,

$$E(\Delta t) = \frac{2}{\sqrt{\pi}} < x > . \tag{B5}$$

A linear fit to $E(\Delta t)$ then gives s used in equation (3).

Alternatively, the variance of the forecast errors is

$$\sigma^2 = \frac{\int_0^\infty (x - <x>)^2 \, e^{-x^2/E^2} 2\pi x dx}{\int_0^\infty e^{-x^2/E^2} 2\pi x dx} \quad \text{(B6)}$$

$$= \frac{\int_0^\infty \left(x^2 - E\sqrt{\pi}x + \frac{\pi}{4}E^2\right) e^{-x^2/E^2} x dx}{E^2/2} \quad \text{(B7)}$$

$$= \left(1 - \frac{\pi}{4}\right) E^2 \; . \quad \text{(B8)}$$

Hence, using the measured standard deviation of the errors at Δt,

$$E(\Delta t) = \frac{\sigma}{\sqrt{1 - \frac{\pi}{4}}} \; . \quad \text{(B9)}$$